Topological Symmetry And Existence of Partial Synchronization

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We study the relationship between the partial synchronous (PaS) state and the coupling structure in general dynamical systems. By the exact proof, we find the sufficient and necessary condition of the existence of PaS state for the coupling structure. Our result shows that the symmetry of the coupling structure is not the equivalent condition which is supposed before but only the sufficient condition. Furthermore, for the existence of the PaS state, the general structure is the equal-degree random.

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An interesting form of dynamical behavior occurs in the coupled systems when only some of the subsystems behave in the same way, which is Partial Synchronization (PaS). The behavior has attracted much attention in the physical [1], biological [2], ecological [3], and other systems. As the coupling strength between the subsystems is small enough, the state of coupled system is turbulence. Increasing the coupling strength, the PaS state is often observed before the Global Complete Synchronization (CS). The existence of various PaS states leads to the remarkably complex dynamical behaviors [4] in the coupled systems.

Considering the dynamics [5]

$$\mathbf{X}_{n+1} = \mathbf{F}(\mathbf{X}_n) + \varepsilon\Gamma \otimes C\mathbf{F}(\mathbf{X}_n), \tag{1}$$

where $\mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N)$ represents the status of the system. The independent state, which is defined as $\mathbf{x}^1 \neq \mathbf{x}^2 \neq \cdots \neq \mathbf{x}^N$, and the CS state, which is defined as $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N$, exist for common coupling ways [6] between the subsystems. However it is possible for a given coupled system without any PaS solutions, Fig. 1 just shows an example. Substituting all possible 365 PaS solutions (e.g., $\mathbf{x}^1 = \mathbf{x}^2$ and $\mathbf{x}^3 \neq \mathbf{x}^4 \neq \mathbf{x}^5 \neq \mathbf{x}^6$) into Eq. (1) at the coupling structure shown in Fig. 1 will give false statements.

In recent years, it is believed there are some tight relations between the topological symmetry of the coupling structure and the PaS state. For example, the asymmetric PaS pattern that disagrees with a symmetrical structure has never been observed [7]; the symmetry group theory can be used to describe the partial periodical synchronous state in some regular structures with the same symmetry [8]; all of PaS states corresponding to each topological symmetry in a ring were observed [9], etc. So one could suppose that the symmetry is the necessary and sufficient condition for the existence of the PaS state.

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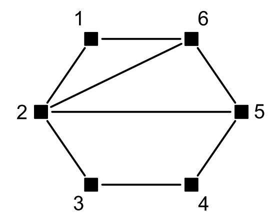

FIG. 1: A topological structure of the coupled system without any partial synchronous solution.

For small amount of subsystems, it seems that the above relationship comes into existence. The coupled structure Fig. 2(a) can be represented mathematically by the adjacent matrix

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Here, node 1 and 2 is symmetric in the structure, and A_4 will be invariable under a permutation transformation $1 \to 2$ or $2 \to 1$. The curves $d_{1,2}(\varepsilon)$, $d_{2,3}(\varepsilon)$ [10] are shown in Fig. 2 (b), ε is the coupling strength. The synchronous solution $\mathbf{x}_1 = \mathbf{x}_2 \neq \mathbf{x}_3 \neq \mathbf{x}_4$ is observed in the region $[0.3, 0.45] \cup [0.7, 1]$ of ε. Thus, the PaS state will be achieved with the corresponding symmetry in A_4 among the symmetrical nodes.

A more complex case is shown in Fig. 3 (a) with the same dynamics as above. There are two clusters and their nodes are denoted by $1, 2, \cdots, n_1$ and $n_1+1, n_1+2, \cdots, n_1+n_2$. Node i $(1 \leq i \leq n_1)$ is coupled to the nodes $i-k, i-k+1, \cdots, i+k, n_1+i-l, n_1+i-l+1, \cdots, n_1+i+l$, and node n_1+i $(1 \leq i \leq n_1)$ is coupled to the nodes n_1+i-k, $n_1+i-k+1, \cdots$, n_1+i+k, $i-l, i-l+1, \cdots, i+l$. Obviously, there is "rotate" symmetry in every cluster. The adjacent matrix will be invariant under a "rotate" permutation transformation in each of them: e.g., it is $1 \to 2$, $2 \to 3$, $\cdots$,

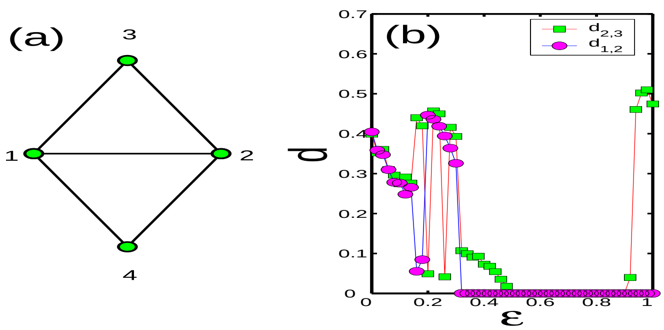

FIG. 2: (a) A simple, but typical, network with symmetries. (b) The average distance $d_{1,2}$ and $d_{2,3}$ versus the coupling strength ε.

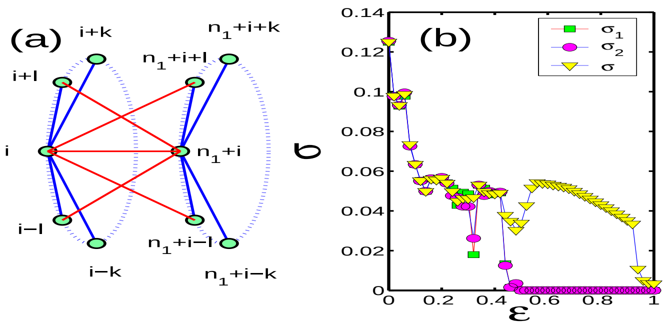

FIG. 3: (a) The scheme of a topological structure with the "rotate" symmetry. (b) The variation of the two clusters σ_1, σ_2 and the whole system σ as functions of ε with $n_1 = n_2 = 100$, $k = 40$, $l = 10$.

$n_1 - 1 \to n_1$, $n_1 \to 1$. We define a $M \times M$ matrix R_M, where $(R_M)_{M,1} = 1$, $(R_M)_{i,i+1} = 1$ $(i = 1, 2, \cdots, M-1)$, and other elements of R_M are 0. Thus the permutation matrix [12] of this transformation will be $T_d = R_{n_1} \oplus R_{n_2}$, where "$\oplus$" is direct sum of two matrices. Fig. 3 (b) shows the time average of the variation of all subsystems $\sigma(\varepsilon) = \lim_{N' \to \infty} \frac{1}{N'} \sum_{n=1}^{N'} \sqrt{\sum_{i=1}^{N} (\mathbf{x}_n^i - \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_n^i)^2}$, and of the two clusters $\sigma_1(\varepsilon)$, $\sigma_2(\varepsilon)$ with $n_1 = n_2 = 100$, $k = 40$, $l = 10$. For $\varepsilon \in [0.45, 1]$, $\sigma_1 = \sigma_2 = 0$, namely $\mathbf{x}_1 = \cdots = \mathbf{x}_{n_1}$, $\mathbf{x}_{n_1+1} = \cdots = \mathbf{x}_N$. The PaS solution of the "rotate" symmetrical nodes is observed.

In this Letter, we investigate in detail the relationships between the PaS solution and the coupling structure. The PaS solution is defined as follow: for a dynamical system with phase space $\mathbb{R}^{Nm}$, a K-cluster synchronous solution is a K-dimensional subspace, denoted by V, of $\mathbb{R}^{Nm}$. It can be represented by

$$\begin{array}{c} \mathbf{x}^{i_1^1} = \mathbf{x}^{i_1^2} = \cdots = \mathbf{x}^{i_1^{n_1}} \\ \mathbf{x}^{i_2^1} = \mathbf{x}^{i_2^2} = \cdots = \mathbf{x}^{i_2^{n_2}} \\ \cdots\cdots \\ \mathbf{x}^{i_K^1} = \mathbf{x}^{i_K^2} = \cdots = \mathbf{x}^{i_K^{n_K}} \end{array} \qquad (2)$$

where $\mathbf{x}^{i_a^b}$ denotes the bth subsystem in the ath cluster and $\{n_i\}_{i=1}^{K}$ is the size of each cluster that satisfied $\sum_{i=1}^{K} n_i = N$ $(N > K > 1)$. The CS and independent solution are the particular cases of the definition for $K = 1$ and $K = N$. The relationship between PaS solution and the coupling structure could be described by two questions as follows:

Question A: If one finds symmetry in a coupling structure, can a corresponding PaS solution be obtained?

Considering the matrix form of Eq. (2),

$$T\mathbf{X} = \mathbf{X}, \forall \mathbf{X} \in V, \qquad (3)$$

where T is a permutation matrix. $\mathbf{X} \in V$ is invariant under the transformation T, so V is the invariant subspace of T and the eig-subspace of T corresponding to eigvalue 1. That V is the invariant subspace of the dynamical system Eq. (1) requires

$$C\mathbf{X} \in V, \quad \forall \mathbf{X} \in V. \qquad (4)$$

Next, if there is a symmetry T in structure C, then C will be invariant under a permutation transformation T. The mathematical representation is

$$T^{-1}CT = C, \qquad (5)$$

Therefore, the matrices C and T are commute, or

$$CT = TC. \qquad (6)$$

Question A can be represented by a mathematical statement as follow:

$$\begin{array}{ll} If & T\mathbf{X} = \mathbf{X}, \\ then & Eq.\ (5) \Rightarrow Eq.\ (4) \end{array}$$

Multiplying the two sides of Eq. (6) by $\mathbf{X} \in V$, we have

$$TC\mathbf{X} = CT\mathbf{X}, \forall \mathbf{X} \in V; \qquad (7)$$

Combining Eq. (3) with Eq. (7) gives

$$TC\mathbf{X} = C\mathbf{X}, \forall \mathbf{X} \in V. \qquad (8)$$

So, $C\mathbf{X}$ is also the eigvector of T with eigvalue 1. And then, Eq. (4) will be satisfied for all $\mathbf{X} \in V$. Thus, we conclude that, for a symmetrical structure, the dynamical system has a corresponding PaS solution.

Question B: If one finds a PaS solution, can the corresponding symmetry in the coupling structure be observed.

Here, an interesting example is shown in Fig. 4 (a). Considering two clusters, each one has n subsystems, every subsystem is randomly connected to $[pn] + 1$ subsystems in the same cluster (p is a probability and $[pn]$ means the integer part of pn) and $[p_r n] + 1$ subsystems in another cluster (p_r is also a probability). Each subsystem in each cluster has equal degree and the connections between two clusters are also equal-degree. Fig. 4 (b) shows the variance of the two clusters (σ_1, σ_2) and the whole system (σ) as functions of the coupling strength ε in the parameters $n = 100$, $p = 1$, $p_r = 0.5$. As $\varepsilon \in [0.34, 0.72]$,

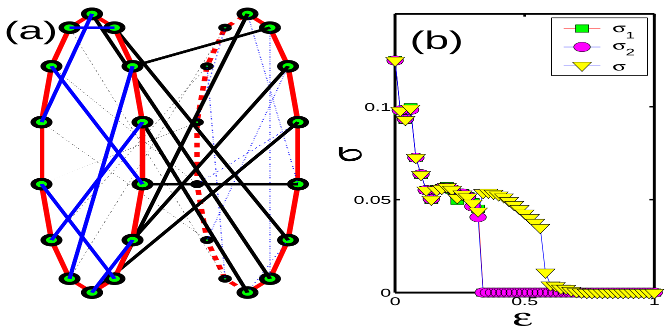

FIG. 4: (a) The scheme of an equal degree random structure. (b) The variance of the two clusters σ_1, σ_2 and the whole network σ as functions of the coupling strength ε in the parameter $n_1 = n_2 = 100$, $p = 0.5$, $p_r = 1$.

$\sigma_1 = \sigma_2 = 0$ and $\sigma > 0$, the PaS solution is observed. Due to the random connections between the subsystems, there is not symmetry in the structure.

Question B can be represented by a mathematical statement as follow:

$$\begin{array}{rl} If & T\mathbf{X} = \mathbf{X}, \\ then & Eq.\ (4) \Rightarrow Eq.\ (5) \end{array}$$

The following relations could be derived

$$\begin{array}{rc} & C\mathbf{X} \in V, T\mathbf{X} = \mathbf{X}, \forall \mathbf{X} \in V \\ \Longleftrightarrow & TC\mathbf{X} = C\mathbf{X} \\ \Longleftrightarrow & TC\mathbf{X} = CT\mathbf{X} \\ \Longleftrightarrow & (T^{-1}CT - C)\mathbf{X} = 0 \\ or & (CT - TC)\mathbf{X} = 0. \end{array} \quad (9)$$

Obviously, Eq. (9) is not equivalent to Eq. (5). We can conclude that it is possible for a PaS solution without any symmetry in a dynamical system. Fig. 4 just gives an example. Another important feature of Eq. (9) is that, in fact, the sufficient and necessary condition can be drawn from Eq. (9).

The component form of Eq. (9) is

$$\sum_{n=1}^{N} C_{mn}\mathbf{x}^n - \sum_{j=1}^{N} C_{ij}\mathbf{x}^j = 0, \forall \mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N) \in V. \quad (10)$$

for $T_{im} = 1$. Relabelling the subsystems in order to collect together the subsystems in the same cluster, Eq. (2) will be rewritten in the form

$$\begin{array}{c} Cluster 1: \\ \mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^{n_1} \equiv \mathbf{y}^1 \\ \cdots\ \cdots \\ Cluster s: \\ \mathbf{x}^{\sum_{k=1}^{s-1} n_k + 1} = \mathbf{x}^{\sum_{k=1}^{s-1} n_k + 2} = \cdots = \mathbf{x}^{\sum_{k=1}^{s-1} n_k + n_s} \equiv \mathbf{y}^s \\ \cdots\ \cdots \\ Cluster K: \\ \mathbf{x}^{\sum_{k=1}^{K-1} n_k + 1} = \mathbf{x}^{\sum_{k=1}^{K-1} n_k + 2} = \cdots = \mathbf{x}^{\sum_{k=1}^{K-1} n_k + n_K} \equiv \mathbf{y}^K. \end{array} \quad (11)$$

And the general form [13] of T will be

$$T = \bigoplus_{i=1}^{K} R_{n_i}, i = 1, 2, \cdots, K. \quad (12)$$

Then the general form of $\mathbf{X}$ will be $\mathbf{X} = (\mathbf{y}^1 1_{1,n_1}, \mathbf{y}^2 1_{1,n_2}, \cdots, \mathbf{y}^K 1_{1,n_K})^T$ where $1_{M,N}$ is an $M \times N$ matrix in which every element is 1. Eq. (10) is the ith row of Eq. (9) and $\{y^s\}_{s=1}^K$ are independent, thus we can collect Eq. (10) into K terms and the sth ($s = 1, 2, \cdots, K$) term describes the degree of subsystem i contributed by sth cluster. If subsystem i belongs to cluster s', then the sth term will be

$$\mathbf{y}^s \sum_{j=N_s+1}^{N_s+n_s} (C_{ij} - C_{mj}) = 0, (i = N_{s'}+1, N_{s'}+2, \cdots, N_{s'}+n_{s'}) \quad (13)$$

where $N_s = \sum_{i=1}^{s-1} n_i$ and $m = i + 1$ when $i < N_{s'} + n_{s'}$; $m = 1$ when $i = N_{s'} + n_{s'}$.

Considering two different situations in Eq. (13), we can draw two statements as follow:

S1: $s \neq s'$. $\sum_{j=N_s+1}^{N_s+n_s} C_{ij}$ is the degree of subsystem i, which is contributed by the sth cluster. So Eq. (13) shows that **the degrees of subsystems in a cluster** (e.g., the s'th cluster), **which are contributed by the subsystems in another cluster** (e.g., the sth cluster), **should be the same.**

S2: $s = s'$. Since $C_{mm} = C_{ii} = -1$, the rest part of Eq. (13) illustrates that **the subsystems' degrees contributed by their cluster also should be the same**.

The two statements are the complete representation of Eq. (9) and now we have a clear physical picture of the necessary and sufficient condition of the existence of a PaS solution. Then, by using the two statements in the particular cases, we can obtain some interesting results:

I. The nonexistence of any PaS states in the system shown in Fig. 1 is easy to be proved. Now we do not need to substitute all the possible solutions into the dynamical system. One can suppose that there is at least one PaS solution. According to **S1** and **S2**, the PaS should be observed between the subsystems with the same degrees, so the subsystem 2 itself must be a PaS cluster without anyone else. On the other hand, subsystem 2 is connected to subsystems 1, 3, 5, 6. **S1** requires that these four subsystems should be included in one or more clusters, but any of these clusters will not include subsystem 4, because the degree of subsystem 4 contributed by node 1 is 0 while that of others is 1, i.e., subsystem 4 itself should be a cluster too. And then, subsystem 4 is connected to subsystem 3 and 5, and because of the same reason, subsystem 3, 5 should be in different clusters with subsystem 1, 6. However, the degrees of subsystem 3 and subsystem 1 are different with that of subsystem 5 and subsystem 6 respectively, that means they also should be in different clusters. So, we can see there will be at least 6 clusters in this system with just 6 subsystems, any PaS phenomenon will not be observed in this system.

II. If there is symmetry in a dynamical system, Eq. (5) can be rewritten as

$$C_{ij} = C_{mn}, \tag{14}$$

where $i = N_{s'}+1,\ N_{s'}+2, \cdots, N_{s'}+n_{s'}$; $j = N_s+1,\ N_s+2, \cdots, N_s+n_s$, $N_s = \sum_{i=1}^{s-1} n_i$ and $m = i+1$ when $i < N_{s'}+n_{s'}$; $m = 1$ when $i = N_{s'}+n_{s'}$; $n = j+1$ when $j < N_{s'}+n_{s'}$; $n = 1$ when $j = N_{s'}+n_{s'}$. So Eq. (14) is a stronger condition than Eq. (13). That is why in few body systems [7, 8, 9] there are tight relations between the PaS solution and the symmetry in the coupling structure. Furthermore, one can prove that if there are only one or two subsystems in every cluster ($n_{s'}, n_s = 1$ or $2,\ s, s' = 1, 2, \cdots, K$) then there should be at least one symmetry in the coupling structure ($n_{s'}, n_s = 2$ is non-trivial). In this case, the $(C_{ij})_{n_{s'}\times n_s}$ can only be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and Eq. (14) will be satisfied automatically.

III. The CS, as a special case of the PaS, can also be included in the Eq. (9), where the dimension of V is only 1. And its normalized basis vector is $\mathbf{X}_{global} = 1_{N,1}/\sqrt{N}$. In fact, this vector is always the eigvector of all of T. because every PaS subspace contains the CS subspace ($\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N$). The C is a non-row matrix (the summery of any row is 0) and the matrix product TC, CT also are. Thus, we conclude that, for the coupled system Eq. (1), the CS solution always exists no matter what the coupling structure C is. So Eq. (9) can be satisfied for all of C in this case, i.e., the CS solution exists in every system Eq. (1) no matter what the system topology C is.

In conclusion, we have studied the relationship between the coupling structure and the PaS state in general dynamical systems. The sufficient and necessary condition of the existence of PaS state for the coupling structure is found by the exact proof. And the result is counterintuitive, that is, the existence of PaS state doesn't require the symmetry in the coupling structure, which is supposed before. Then, a new structure, the equal-degree random structure, is obtained. According to the sufficient and necessary condition, it is the general structure for the existence of the PaS state. Finally, we should stress that the proof also can be applied to the differential dynamical systems and the conclusions are the same.

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[5] This is one of widely used forms of the coupling nonlinear maps. One can apply certain m-dimensional map $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, $\mathbf{x} = (x_n^1, x_n^2, \cdots, x_n^m)$ as subsystem to construct a coupled system $\mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N)$, $\mathbf{F} = (\mathbf{f}^1, \mathbf{f}^2, \cdots, \mathbf{f}^N)$. The common interesting is the linear coupling between the subsytems, thus the coupling term will be $\varepsilon\Gamma \otimes C\mathbf{F}(\mathbf{X})$, where ε the coupling strength. $\Gamma : \mathbb{R}^m \to \mathbb{R}^m$ characterizes the coupling scheme among the subsystems and C is the Laplacian matrix. $C_{ij} = A_{ij}/k_i$ for $j \neq i$ and $C_{ii} = -1$, where k_i is the degree of subsystem i and A_{ij} is an element of the adjacent matrix A of the coupling structure. To gain insight in numerical simulation, we set $f(x)$ to be the logistic map $f(x) = 4x(1-x)$, but the proof do not depend on the dynamics.

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[13] The T_d in the second example is only a special case of Eq.(12), and the permutation transformation done to A_4 can also be represented by $T_4 = R_2 \oplus 1 \oplus 1$.